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### **Waiting Times When Service Times are Stable Laws: Tamed and Wild**

by

Donald P. Gaver  
Patricia A. Jacobs

August 1996

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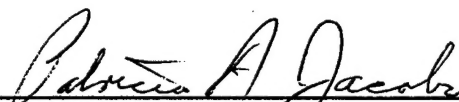
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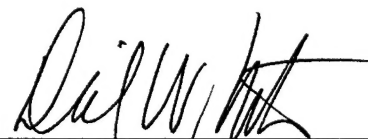
  
DONALD P. GAVER  
Professor of Operations Research

  
PATRICIA A. JACOBS  
Professor of Operations Research

Reviewed by:

Released by:

  
FRANK PETHO  
Chairman  
Department of Operations Research

  
DAVID W. NETZER  
Associate Provost and Dean of Research

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# WAITING TIMES WHEN SERVICE TIMES ARE STABLE LAWS: TAMED AND WILD

Donald P. Gaver

Patricia A. Jacobs

Department of Operations Research  
Naval Postgraduate School  
Monterey, CA 93943

## Abstract

Modern telecommunication systems must accommodate tasks or messages of extremely variable time duration. Understanding of that variability, and appropriate stochastic models are needed to describe the resulting queues or buffer contents. To this end, consider an M/G/1 queue with service times having a positive stable law distribution. Such service times are extremely long (and short) tailed, and thus do not have finite first and second moments; classical queue-theoretic results do not apply directly. Here we suggest two procedures for initially *taming* stable laws, i.e. so that they possess finite mean and variance. We apply the tamed laws to calculate certain familiar queuing properties, such as the transform of the stationary distribution of the long-run virtual waiting time and mean thereof. We show that, by norming or scaling traffic intensity, waiting times, and other measures of congestion, we can obtain *bona fide* limiting distributions as the underlying service times become untamed, i.e. return to the wild. Simulations support the theory.

## 1. The Problem Motivation

In various applications of service system or queuing theory there may arise a need to consider service times,  $S$ , of great variability, i.e. that seem to possess nearly Pareto tails:

$$P\{S > x\} \equiv 1 - F_S(x) = O(x^{-\alpha}) \quad (1.1)$$

as  $x \rightarrow \infty$ , where  $\alpha$  is small enough so that no moments,  $E[S^k]$ ,  $k \geq 1$ , are finite. In this paper we examine certain aspects of such problems for M/G/1 systems, focusing on service times that are describable by *positive stable laws*. In view of Theorem 1 on p. 448, Feller, II (1971), it is impossible to ignore the class of stable law models to represent the behavior of (1.1); there is the additional fact that stable laws approximate the distributions of sums of many long-tailed independent random variables, e.g. the sum of a number of activities that constitute service. But there is the problem that without finite first and second moments at a minimum classical queue-theoretic results do not directly apply.

In this paper we consider some procedures for *taming* stable laws so that they do possess the required properties, i.e. finite moments. We apply the tamed laws to calculate certain familiar queuing properties, such as the virtual waiting time in the system. Then we show that, by norming or scaling waiting times and other measures of congestion, we can obtain bona-fide limiting distributions as the underlying service times become untamed, or “return to the wild”. For similar work see Abate, *et al.* (1993, 1994) and probably more recent articles as well.

The authors are very much indebted to Walter Willinger for pointing out many interesting references attesting to the appearance of long-tailed distributions in modern communications systems. This in no way implicates W. Willinger in our present machinations. We also gratefully acknowledge the work by Ward Whitt and Joe Abate.

## 2. How to Tame a Wild Stable Law

There are several approaches that naturally suggest themselves for endowing a stable-law distributed service time,  $S$ , of scale parameter,  $v$ , and order  $0 \leq \alpha < 1$ , with finite moments. Recall from Feller (1971) that the Laplace-Stieltjes transform of  $S$  is

$$E[e^{-sS}] = \exp[-(vs)^\alpha]. \quad (2.1)$$

For the value  $\alpha = 1/2$  the above possesses an explicit inverse, the name of which is *inverse Gaussian*, a slight misnomer since the distribution is actually that of the inverse square root of a Gaussian. Otherwise inverses are only expressible as unintelligible infinite series or the equivalent.

## (2.1) Assessing Shape Indirectly

As stated, (wild) stable laws possess no finite moments of order  $\geq 1$ . Furthermore, there are no conveniently obtained explicit quantiles (e.g. median, lower or upper quantiles, etc.) of the above, but there are simple substitutes based on exponential distributions: ask for the *test* or *killing* exponential density, of mean  $\kappa(p)$ , an observation from which,  $X$ , exceeds  $S$  with probability  $p$ . We get from (2.1)

$$P\{S < X\} = \exp\left[-(v/\kappa(p))^\alpha\right] = p \quad (2.2)$$

or

$$\kappa(p) = v[\ln(1/p)]^{-1/\alpha}.$$

It is seen that the *exponential median*,  $\kappa(1/2)$ , approaches  $\infty$  as  $\alpha \rightarrow 0$ , and approaches  $v/\ln(2)$  as  $\alpha \rightarrow 1$ , not surprisingly since for  $\alpha = 1$  the value of  $S = v$  with probability 1. The value  $p = e^{-1} = 0.368$  is pivotal: for  $p = e^{-1}$ ,  $\kappa(e^{-1}) = v$  for all  $\alpha$ ; for  $p < e^{-1}$  ( $1/p > e$ )  $\kappa(p)$  increases with  $\alpha \uparrow 1$ ; for  $p > e^{-1}$  ( $1/p < e$ )  $\kappa(p)$  decreases as  $\alpha \uparrow 1$ .

## (2.2) I, Taming by Tilting: Initial Screening

Large deviation theory exploits an exponential tail by positive tilting towards large values of interest so that the central limit theorem can be applied. Here it is useful to apply *negative tilting*, see Abate *et al.*, (1995) and (1994); they call this

*exponential damping*, while we speak of *taming*. Look at  $S$ -values that terminate before being killed:

$$P\{S_\kappa \leq t\} = P\{S \leq t | S \leq X\} = \int_0^t f_S(x) e^{-x/\kappa} dx / \int_0^\infty f_S(x) e^{-x/\kappa} dx \quad (2.3)$$

which has transform

$$E[e^{-sS_\kappa}] = \exp\left[-(v(s+1/\kappa))^\alpha\right] / \exp\left[-(v/\kappa)^\alpha\right]. \quad (2.4)$$

In a queuing context the above might arise naturally as a control strategy:  $\kappa$ , the mean of the service-killing distribution, is selected so as to keep the sizes of the jobs selected under control. The subset of jobs that pass the exponential killing screen are actually allowed into service, so if  $\lambda$  is the arrival rate then the system only sees  $\lambda \cdot e^{-(v/\kappa)^\alpha}$  as arrival rate, and (unscaled) traffic intensity is

$$\begin{aligned} \rho(\kappa) &= \lambda P\{S < X\} E[S | S < X] \\ &= \lambda e^{-(v/\kappa)^\alpha} \cdot \alpha (v/\kappa)^{\alpha-1} v \\ &\sim \lambda \alpha v^\alpha \kappa^{1-\alpha} \rightarrow \infty \end{aligned} \quad (2.5)$$

as  $\kappa \rightarrow \infty$ . Only if  $\lambda \kappa^{1-\alpha} = O(1)$  is there hope of achieving a steady-state distribution.

### (2.3) II, Taming by Truncation: On-Line Completion, Perhaps Partial

Suppose that each time an  $S$ -value is realized a killing (or interruption) value  $X$  is independently realized. Total service is rendered if the service survives, i.e.  $S < X$ ; otherwise partial service  $X < S$  is rendered and a new job can be accepted as soon as either event occurs. This setup can be called on-line real-time killed service. For the server, it means that the effective service time is  $\tilde{S}_\kappa = \min(S, X)$ , with transform



$$\begin{aligned}
E\left[e^{-s\tilde{S}_\kappa}\right] &= \int_0^\infty e^{-sx} e^{-x/\kappa} f_S(x) dx + \int_0^\infty e^{-sx} (1 - F_S(x)) e^{-x/\kappa} 1/\kappa dx \\
&= e^{-(\nu(s+1/\kappa))^\alpha} + \frac{1 - e^{-(\nu(s+1/\kappa))^\alpha}}{1 + \kappa s} \\
&= \frac{1}{1 + \kappa s} + \frac{\kappa s e^{-(\nu(s+1/\kappa))^\alpha}}{1 + \kappa s}.
\end{aligned} \tag{2.6}$$

Now

$$\begin{aligned}
E[\tilde{S}_\kappa] &= \kappa \left(1 - e^{-(\nu/\kappa)^\alpha}\right) \\
&\sim \nu^\alpha \kappa^{1-\alpha} \rightarrow \infty
\end{aligned} \tag{2.7}$$

as the mean killing time  $\kappa \rightarrow \infty$ , so again only if  $\lambda \kappa^{1-\alpha} = O(1)$  will there be an opportunity for long-run queue stability.

### 3. Transforms of Long-Run Waiting Times

The formula for the Laplace-Stieltjes transform of the long-run or steady-state distribution of M/G/1 virtual waiting time,  $W$ , is well known to be

$$E\left[e^{-sW}\right] = \frac{1 - \rho}{1 - \rho \left( \frac{1 - E[e^{-sS}]}{sE[S]} \right)} \tag{3.1}$$

provided  $\rho = \lambda E[S] < 1$ ; otherwise no such distribution exists and the waiting time tends to increase. Now suppose we contemplate an M/G/1 system with stable law service, tamed as in I or II above, i.e. with exponential,  $X$ , screening or truncating exponentials, such that  $E[X] = \kappa = 1/\mu$ . Then consider a sequence of such, as  $\kappa \rightarrow \infty$  or  $\mu \rightarrow 0$ . We show how to adjust the arrival rate and normalize the waiting time so as to obtain (transforms of) *bona fide* limiting distributions for the normalized virtual waiting times.

First address the scaling of arrival rate  $\lambda$  to control the traffic intensity  $\rho$ . From (2.5) and (2.7) it is necessary that the actual arrival rate becomes small as the taming parameter  $\kappa$  becomes large if the resulting traffic intensity is to be bounded. Therefore take the adjusted arrival rate to be  $\lambda^* = \lambda \kappa^{1-\alpha}$  constant; the constant is chosen so that the relevant traffic intensity is less than 1. For the screening situation, *I*,

$$\rho_I^* = \lambda^* e^{-(v/\kappa)^\alpha} \alpha v^\alpha \sim \lambda^* \alpha v^\alpha \quad (3.2)$$

and for the truncation situation, *II*,

$$\rho_{II}^* = \lambda^* \kappa^\alpha \left(1 - e^{-(v/\kappa)^\alpha}\right) \sim \lambda^* v^\alpha. \quad (3.3)$$

For particular stable law input *I*-taming results in smaller system load than does *II*-taming since  $\alpha \leq 1$ . This is to be expected, as the latter admits some arrivals that the former rejects outright.

Assuming the above, consider the normalized random variable  $W^* = W/\kappa \equiv W\mu$ . Replace  $s$  by  $\theta/\kappa \equiv \theta\mu$  to obtain

$$E\left[e^{-\theta W^*}\right] = \frac{1 - \rho^*}{1 - \rho^* \left( \frac{1 - E\left[e^{-\theta\mu S}\right]}{\theta\mu E[S]} \right)} \quad (3.4)$$

where  $S$  is tamed and  $\rho^* < 1$ .

### (3.1) Screened Service, I

Substitute (2.4) and the expression  $E[S_\kappa] = e^{-(v/\kappa)^\alpha} \alpha (v/\kappa)^{\alpha-1} v \sim \alpha v^\alpha \mu^{\alpha-1}$  into (3.4). The result is a formula for every screening level  $\kappa = 1/\mu$ . Now take the limit as  $\mu \rightarrow 0$ :

$$E\left[e^{-\theta W_I^*}\right] = \frac{1 - \rho_I^*}{1 - \rho_I^* \left( \frac{(1+\theta)^\alpha - 1}{\alpha\theta} \right)}. \quad (3.5)$$

It is clear from construction and also from directly expanding that

$$\psi(\theta) = \frac{(1+\theta)^\alpha - 1}{\alpha\theta} \quad (3.6)$$

is completely monotone, hence the transform of an honest distribution. By differentiation or otherwise

$$E[W_I^*] = \frac{\rho_I^*}{1-\rho_I^*} \cdot \left( \frac{1-\alpha}{2} \right). \quad (3.7)$$

It is immediately seen that the limiting distribution of the scaled limiting random variable  $W_I^*$  does not depend on  $\nu$ , the original stable law scale, except through the traffic intensity  $\rho^* = \rho_I^* < 1$ .

### (3.2) Truncated Service, II

The effect of on-line service truncation is traced by substituting the transform (2.6) into (3.4). Take the limit as  $\mu \rightarrow 0$ , i.e. untame, to obtain

$$E[e^{-\theta W_{II}^*}] = \frac{1-\rho_{II}^*}{1-\rho_{II}^* \left( \frac{1}{1+\theta} \right)^{1-\alpha}} \quad (3.8)$$

for  $\rho_{II}^* < 1$ . This is recognized to be the transform of a geometric mixture of gammas with scale 1 and shape parameter  $1-\alpha$ . In this case

$$E[W_{II}^*] = \frac{\rho_{II}^*}{1-\rho_{II}^*} (1-\alpha). \quad (3.9)$$

Once again the scaled limiting random variable has a distribution that depends on the service time scale parameter,  $\nu$ , only through the traffic intensity. The fact that  $\rho_I^* = \alpha \rho_{II}^* \leq \rho_{II}^*$  and that a factor of  $1/2$  is present attests to the fact that greater load is placed on system II than on system I. Of course greater service of all incoming arrivals is furnished by II than by I.

#### 4. The Number of Customers in the System

The formula for the generating function of the long-run or steady-state distribution of the number of customers waiting or being served at an arbitrary time in an M/G/1 queue,  $N$ , is known to be

$$E[z^N] = \frac{(1-\rho)(1-z)E[e^{-\lambda(1-z)S}]}{E[e^{-\lambda(1-z)S}] - z} \quad (4.1)$$

where  $S$  is a generic service time and  $\rho < 1$ ; (cf. Gaver [1959]).

##### 4.1 I, Taming by Tilting: Initial Screening

Differentiating the transform of  $S$ , (2.4), and evaluating the results at  $s = 0$  results in

$$E[S] \sim \alpha v^\alpha \kappa^{1-\alpha} \quad (4.2)$$

$$Var[S] \sim \alpha(1-\alpha)v^\alpha \kappa^{2-\alpha} \quad (4.3)$$

as  $\kappa \rightarrow \infty$ . Thus,

$$E[N_I] \sim \frac{\rho_I^{*2}}{2(1-\rho_I^*)} \frac{(1-\alpha)}{\alpha v^\alpha} \kappa^\alpha. \quad (4.4)$$

Substitute (2.4) into (4.1) for  $z = e^{-s/\kappa^\alpha}$

$$\begin{aligned} E[e^{-sN/\kappa^\alpha}] &\sim \frac{(1-\rho_I)\left(\frac{s}{\kappa^\alpha}\right) \exp\left\{-\left[\frac{v}{\kappa}\left[1+\lambda^*s\right]\right]^\alpha + \left(\frac{v}{\kappa}\right)^\alpha\right\}}{\exp\left\{-\left(\frac{v}{\kappa}\left[1+\lambda^*s\right]\right)^\alpha + \left(\frac{v}{\kappa}\right)^\alpha\right\} - e^{-s/\kappa^\alpha}} \\ &\rightarrow \frac{(1-\rho_I^*)}{1-\rho_I^* \left[\frac{[1+\lambda^*s]^\alpha - 1}{\alpha \lambda^* s}\right]} = \frac{(1-\rho_I^*)}{1-\rho_I^* \psi(\lambda^* s)} \end{aligned} \quad (4.5)$$

as  $\kappa \rightarrow \infty$ , where  $\psi$  is defined in (3.6). Note that scaling for  $N$  is by  $\kappa^\alpha$ , while for  $W$  it is by  $\kappa$ . Otherwise (3.5) and (4.5) differ only by a factor  $\lambda^*$  in the denominator.

## 4.2 II, Taming by Truncation

Differentiation of the Laplace transform of the service time (2.6) yields

$$E[S] \sim v^\alpha \kappa^{1-\alpha} \quad (4.6)$$

$$E[S^2] \sim 2(1-\alpha)v^\alpha \kappa^{2-\alpha} \quad (4.7)$$

as  $\kappa \rightarrow \infty$ . Thus,

$$E[N_{II}] \sim \frac{\rho_{II}^{*2}}{(1-\rho_{II}^*)} \frac{(1-\alpha)}{v^\alpha} \kappa^\alpha. \quad (4.8)$$

Note that  $E[N_I] < E[N_{II}]$  as expected.

Substituting (2.6) into (4.1) for  $z = e^{-s/\kappa^\alpha}$ , it follows that

$$\begin{aligned} \lim_{\kappa \rightarrow \infty} E[e^{-sN/\kappa^\alpha}] &= \frac{(1-\rho_{II}^*)}{1 - \left( v \left[ \lambda^* s + 1 \right] \right)^\alpha \frac{\lambda^*}{1+\lambda^* s}} \\ &= \frac{(1-\rho_{II}^*)}{1 - \lambda^* v^\alpha \left[ \lambda^* s + 1 \right]^{\alpha-1}} = \frac{(1-\rho_{II}^*)}{1 - \rho_{II}^* \left( \frac{(1/\lambda^*)}{(1/\lambda^*) + s} \right)^{(1-\alpha)}}; \end{aligned}$$

this is recognizable as the transform of a geometric mixture of gammas with scale  $\frac{1}{\lambda^*}$  and shape parameter  $1-\alpha$ , note its similarity to (3.8). Again the scaling by  $\kappa^\alpha$  is involved.

To date inversion of the transform appearing in the denominator of (3.5) and (4.5) has eluded us. We pose the problem of its inversion, or characterization, to Julian Keilson as a birthday gift. Happy Birthday!

## 5. Busy Periods

It has been seen that normalization by powers of  $\kappa$ , the mean truncation time, permits convergence of the traffic intensity parameter,  $\rho$ , and also the stationary

distribution of virtual waiting time. It is of interest to study the behavior of the busy period when such a normalization is applied. Here convergence to nice distributions does not occur.

Recall that if  $B$  is a busy period duration we can look at its generation in these terms:

$$B = S + B_1 + B_2 + \dots + B_{N(S)} \quad (5.1)$$

where  $S$  is the first service time in the busy period,  $\{B_i, i = 1, 2, \dots\}$  is an iid sequence of copies of busy periods starting with one arrival, and  $N(S)$  is the number of arrivals in  $S$ . By conditional expectation,

$$E[B|S, N(S)] = S + E[B_1]N(S) \quad (5.2)$$

and so

$$E[B] = \frac{E[S]}{1 - \lambda E[S]}. \quad (5.3)$$

If we normalize so that  $\lambda = O(1/\kappa^{1-\alpha})$  as in (3.2) and (3.3) then the traffic intensity tends to a constant as  $\kappa$  increases. It follows from (5.3) above that the expected busy period is  $E[B]$ , like  $E[S]$ , of order  $\kappa^{1-\alpha}$ . This gives hope that the actual distribution of a scaled random busy period,  $B^\# = B/\kappa^{1-\alpha}$ , might converge to some recognizable honest form. However, such does not seem to occur. For positive  $\theta$ ,  $\varphi_B(\theta)$  is the (smallest positive) root of

$$E[e^{-\theta B}] \equiv \varphi_B(\theta) = u[\theta + \lambda(1 - \varphi_B(\theta))] \quad (5.4)$$

where  $u(\theta)$  is the Laplace-Stieltjes transform of the service time. Hence, the normalized busy period would satisfy, for Model I,

$$\varphi_B^\#(\theta) = \exp \left[ - \left( v \left( \frac{\theta + \lambda^*}{\kappa^{1-\alpha}} (1 - \varphi_B^\#(\theta)) + \frac{1}{\kappa} \right)^\alpha \right) \right] \exp \left[ (v/\kappa)^\alpha \right]. \quad (5.5)$$

Differentiation once at  $\theta = 0$  shows that

$$E[B^\#] \sim \frac{\alpha v^\alpha}{1 - \rho_I^*}; \quad (5.6)$$

a similar result, again finite, holds for Model II. However, further analysis shows that for Model I

$$\text{Var}[B^\#] \sim \frac{\alpha(1-\alpha)v^\alpha \kappa^\alpha}{(1 - \rho_I^*)^3} \quad (5.7)$$

i.e. is unbounded as  $\kappa \rightarrow \infty$ , even though it has been normalized and the normalized mean is finite. Similarly, for Model II

$$\text{Var}[B^\#] \sim \frac{2(1-\alpha)v^\alpha \kappa^\alpha}{(1 - \rho_{II}^*)^3} \quad (5.8)$$

which also becomes large like  $\kappa^\alpha$  but remains larger than the previous variance because of more permissive job entry. Recall that the traffic intensities in (5.7) and (5.8) differ; refer to (3.2) and (3.3). In summary, it does not appear possible to scale stable-law-service busy periods so as to achieve a non-zero mean and yet get an honest limiting distribution with finite second moment. This is not surprising in light of the fact that the virtual waiting time must be scaled to obtain such a limit. Nevertheless some qualitative information may be deduced about aspects of system behavior from the likes of (5.7) and (5.8).

## 6. Simulation Results

In this section we describe a simulation experiment and its results.

We consider an M/G/1 queue; the service times have an inverse Gaussian distribution that is tamed by truncation. The transform of the untamed distribution is (2.1) with  $\alpha = 1/2$  and  $v = 2$ . The tamed-by-truncation service time is simulated by

$$S = \min\left(\frac{1}{Z^2}, Y\right)$$

where  $Z$  is a standard normal random variable and  $Y$  is an exponential random variable having mean  $\kappa$ .

The customer arrival rate is determined as follows. Set  $\lambda^*$  in (3.3) equal to 0.8. Put the arrival rate of customers

$$\lambda = \lambda^* / \kappa \left(1 - e^{-(2/\kappa)^{1/2}}\right).$$

The waiting times for successive customers are obtained by recursion

$$W_{n+1} = \max(W_n + S_n - A_{n+1}, 0)$$

where  $W_n$  is the  $n^{\text{th}}$  customer's waiting time in queue,  $S_n$  is the length of the  $n^{\text{th}}$  customer's service time, and  $A_{n+1}$  is the time between the  $n^{\text{th}}$  and  $(n+1)^{\text{st}}$  arrival. Start at  $W_1 = 0$ . Clearly the above does not simulate *virtual* waiting times, but in the case of Poisson arrivals the long-run limiting results are equivalent.

Graphical displays of the time series of simulated waiting times appear in Figures 1a–1e. One is struck by the large variability in the waiting times: upward surges appear to occur occasionally, prevail for awhile, and then be interrupted by periods of rather small but fluctuating values. Even averages of 15,000 in single realizations are not especially stable: the five quoted range, after normalization by  $\kappa = 150$ , from 1.48 to 2.46.

Summaries of 5 replications of the simulation appear in Table 1. In each replication the waiting times for 15,000 customers are simulated, and the waiting times for all customers then averaged. These results are reported: the normed-by- $1/\kappa$  averages per replication, when averaged, turn out to equal 2, with standard error of 0.17. This is in excellent agreement with the result of the theory (3.9), which predicts a value of 2.



TABLE 1  
LIMITING ( $\kappa$ LARGE) SIMULATION  
Mean Waiting Time (Scaled)  
15000 Waiting Times per Replication  
(TAMING II)  
 $\kappa = 150$

	1	2	3	4	5	Mean of Means	Standard Error: $\sqrt{\text{Var}/5}$
Mean	317.0	369.5	324.1	221.4	269.1	300.2	25.3
$\frac{\text{Mean}}{\kappa}$	2.11	2.46	2.16	1.48	1.79	2.00	0.17

## 7. Discussion

Taming, as described above, may be viewed as a control strategy. It could be of interest to ask about the fate of those jobs that are rejected (Model I), or partially finished (Model II): these or their residues, respectively, could be shunted to another server that must handle such overflowing extremely long jobs; presumably these occur at a low enough rate to be accommodated because they are filtered from the mainstream of arrivals. Several such stages could be envisioned, and an attempt made to optimize with respect to the taming or truncation parameters  $\kappa_s$  at stages  $s = 1, 2, \dots$ . In practice a deterministic truncation time would be realistic, but the mathematics is less tractable.

Finally, we point out that Pareto-tailed distributions are not the most pathologically long-tailed possible. A simple option is to mix one positive stable law with another: replace the parameterization (2.1) by  $v^\# = v_1^\alpha$ , where  $v^\#$  is itself stable. The result is expressible as the Laplace transform of the mixing distribution.

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SUCCESSIVE CUSTOMER WAITING TIMES:KAPPA=150  
INVERSE GAUSSIAN DISTRIBUTION;TAMING BY TRUCATION  
REPLICATION 1  
MEAN WAITING TIME=317.0  
MEAN÷KAPPA=2.11

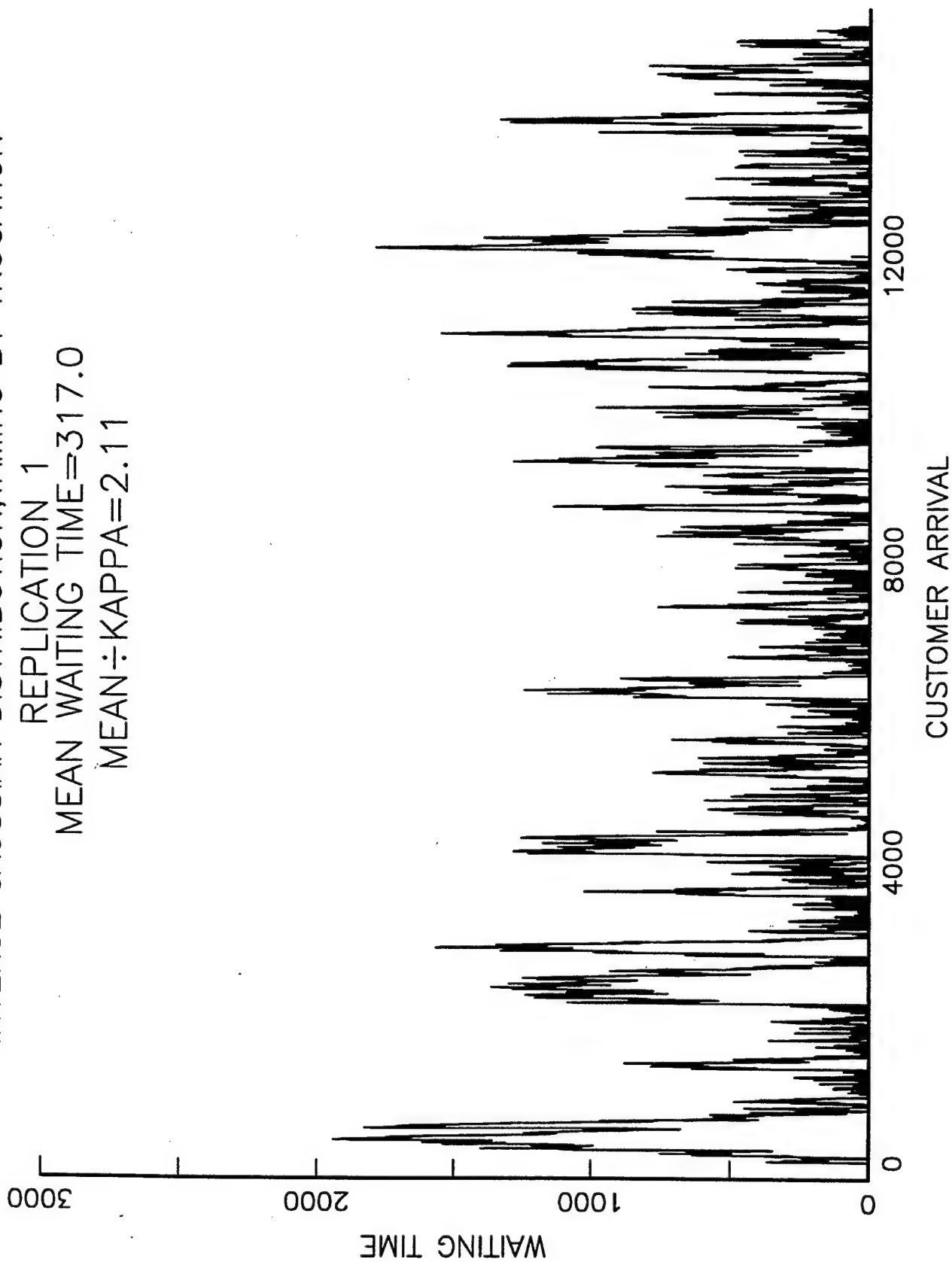


Figure 1a  
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SUCCESSIVE CUSTOMER WAITING TIMES:KAPPA=150  
INVERSE GAUSSIAN DISTRIBUTION;TAMING BY TRUCATION  
REPLICATION 2  
MEAN WAITING TIME=369.5  
MEAN÷KAPPA=2.46

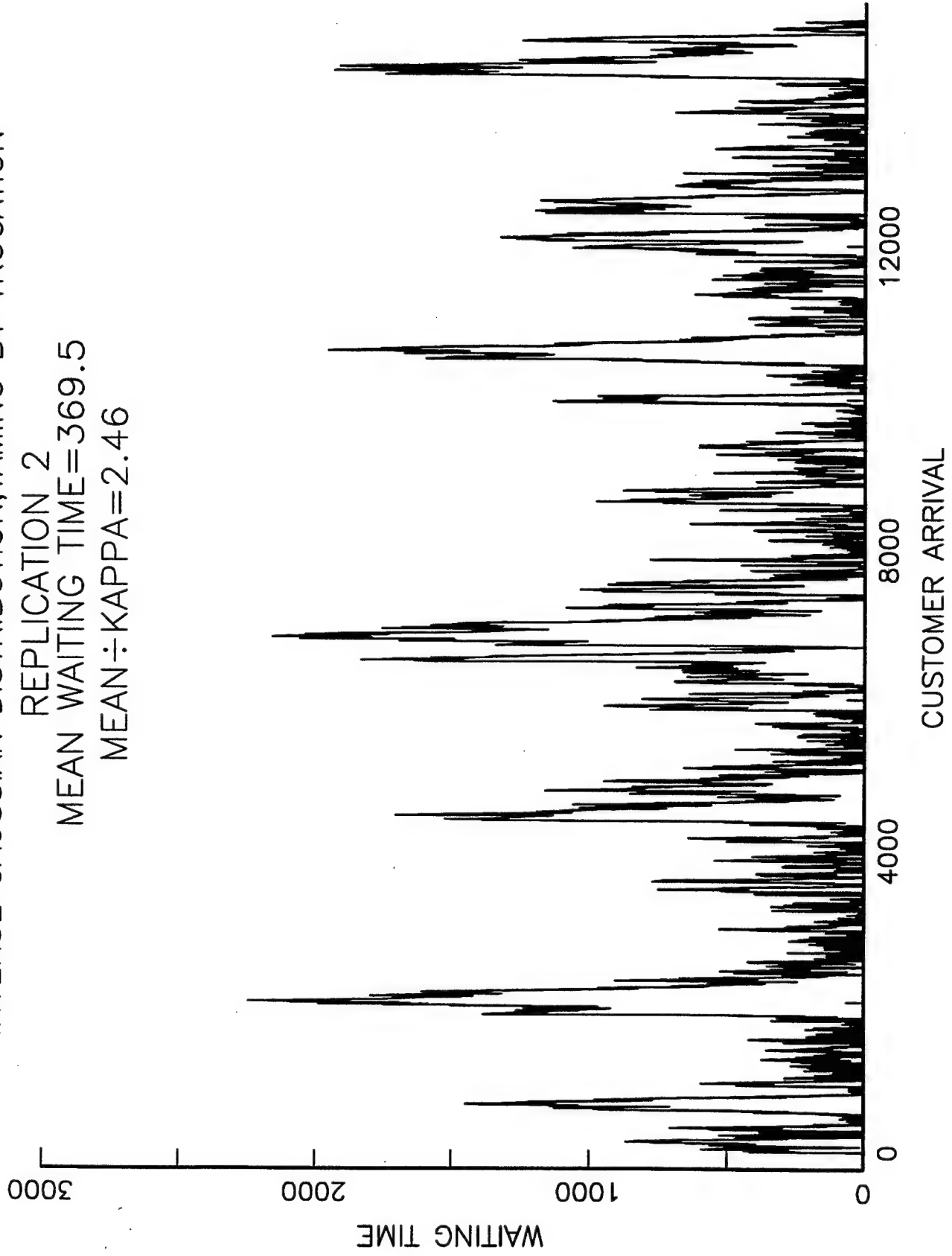


Figure 1b  
16

SUCCESSIVE CUSTOMER WAITING TIMES:KAPPA=150  
INVERSE GAUSSIAN DISTRIBUTION;TAMING BY TRUCATION  
REPLICATION 3  
MEAN WAITING TIME=324.1  
MEAN÷KAPPA=2.16

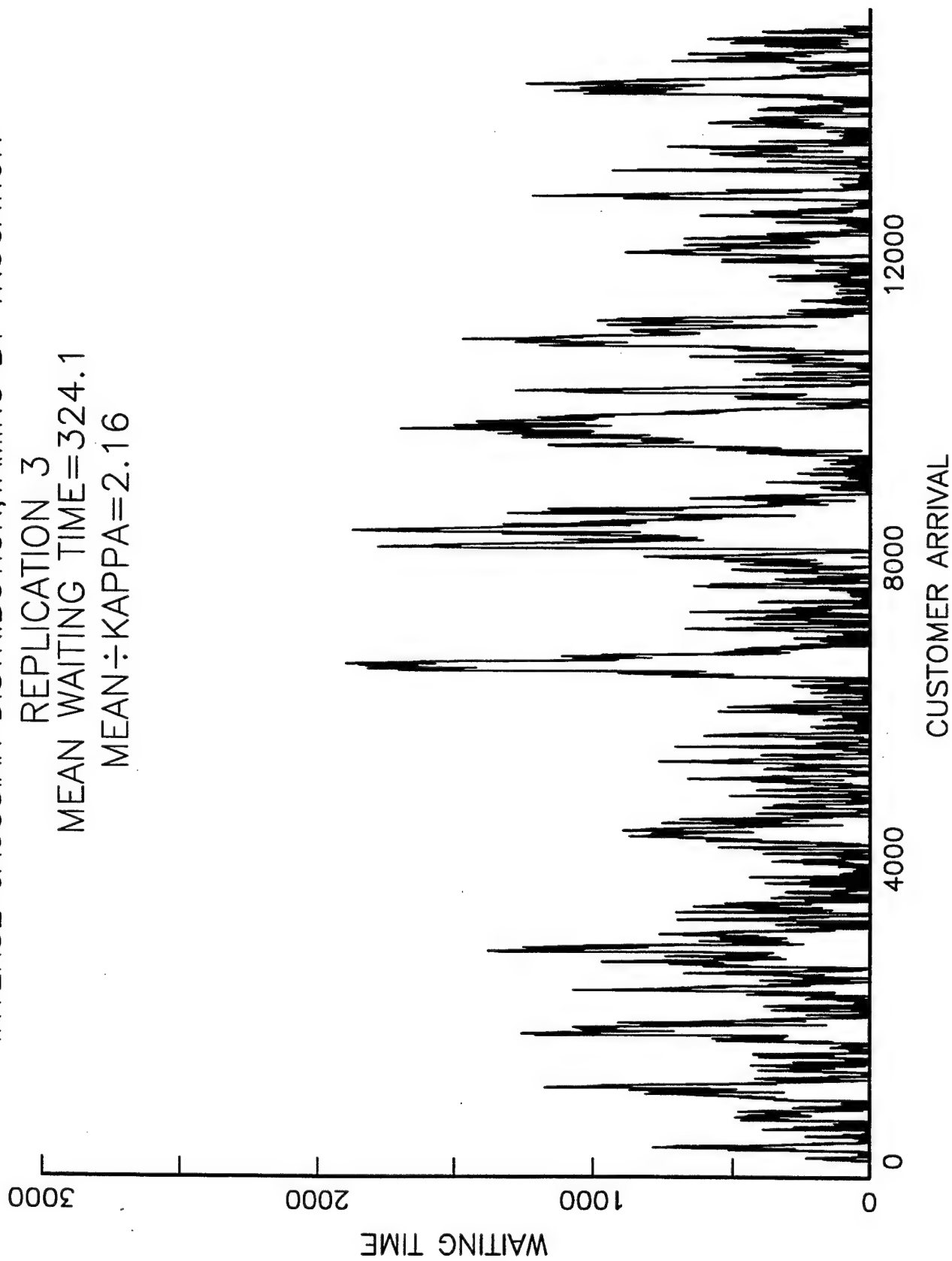


Figure 1c  
17

SUCCESSIVE CUSTOMER WAITING TIMES:KAPPA=150

INVERSE GAUSSIAN DISTRIBUTION;TAMING BY TRUCATION

REPLICATION 5

MEAN WAITING TIME=269.1

MEAN÷KAPPA=1.79

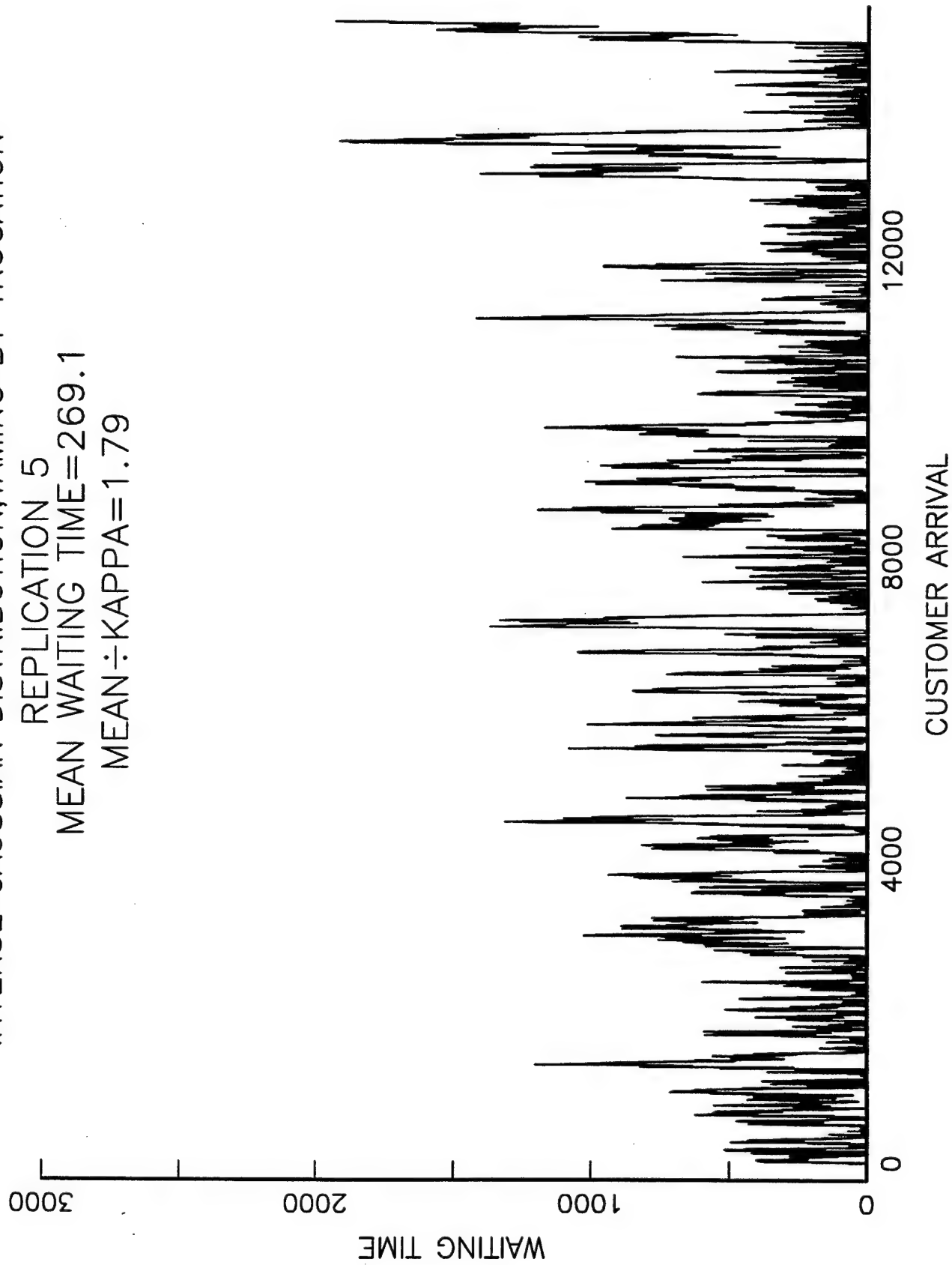


Figure 1d  
18

SUCCESSIVE CUSTOMER WAITING TIMES:KAPPA=150

INVERSE GAUSSIAN DISTRIBUTION;TAMING BY TRUCATION

REPLICATION 4

MEAN WAITING TIME=221.4

MEAN÷KAPPA=1.48

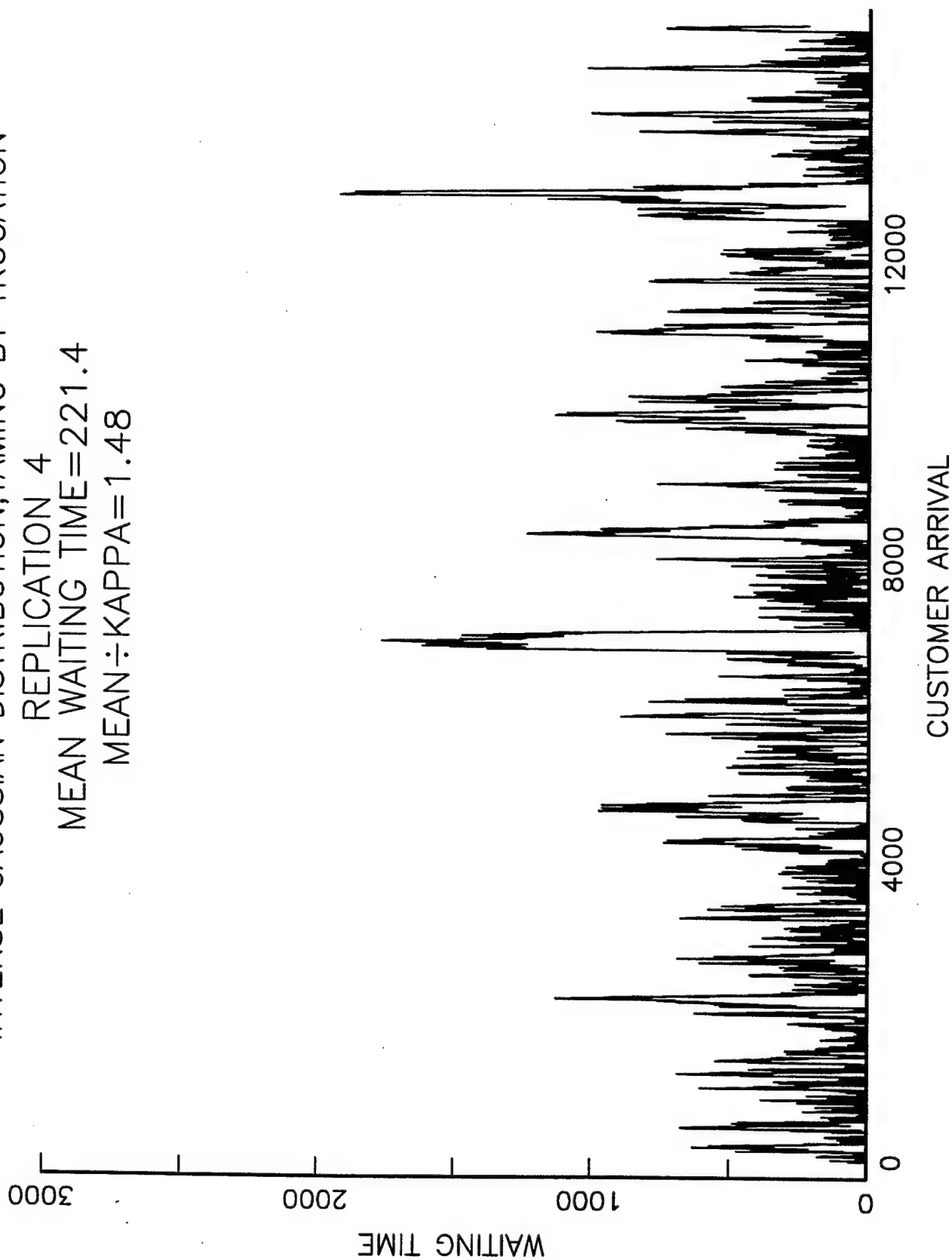


Figure 1e  
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